

Dirac's Equation in R -Minkowski Spacetime

T. Foughali * and A. Bouda†

Laboratoire de Physique Théorique, Faculté des Sciences Exactes,
Université de Bejaia, 06000 Bejaia, Algeria

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Abstract

We recently constructed the R -Poincaré algebra from an appropriate deformed Poisson brackets which reproduce the Fock coordinate transformation. We showed then that the spacetime of this transformation is the de Sitter one. In this paper, we derive in the R -Minkowski spacetime the Dirac equation and show that this is none other than the Dirac equation in the de Sitter spacetime given by its conformally flat metric. Furthermore, we propose a new approach for solving Dirac's equation in the de Sitter spacetime using the Schrödinger picture.

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1 Introduction

Deformed Special Relativity (DSR) [1, 2, 3, 4] and Fock's transformation [5] are two distinct approaches of nonlinear relativity which are developed for completely different motivations. In addition to the speed of light, DSR keeps invariant a minimal length on the order of the Planck length, while Fock's transformation keeps invariant a length which represents the universe radius.

By following the same way as in DSR [6], we recently proposed an appropriate deformed Poisson brackets [7]

$$\{x^\mu, x^\nu\} = 0, \quad (1)$$

$$\{x^\mu, p^\nu\} = -\eta^{\mu\nu} + \frac{1}{R}\eta^{0\nu}x^\mu, \quad (2)$$

$$\{p^\mu, p^\nu\} = -\frac{1}{R}[p^\mu\eta^{0\nu} - p^\nu\eta^{\mu 0}], \quad (3)$$

from which we reproduced the Fock coordinate transformation

$$t' = \frac{\gamma(t - ux/c^2)}{\alpha_R}, \quad x' = \frac{\gamma(x - ut)}{\alpha_R}, \quad y' = \frac{y}{\alpha_R}, \quad z' = \frac{pz}{\alpha_R}, \quad (4)$$

where

$$\alpha_R = 1 + \frac{1}{R} \left[(\gamma - 1)ct - \gamma \frac{ux}{c} \right], \quad (5)$$

*E-mail: fougto_74@yahoo.fr

†E-mail: bouda_a@yahoo.fr

R is the universe radius, $\gamma = (1 - u^2/c^2)^{-\frac{1}{2}}$, $\eta^{\mu\nu} = (+1, -1, -1, -1)$ and $\mu, \nu = 0, 1, 2, 3$. Relations (4) define the so-called R -Minkowski spacetime. From the above brackets, we established the corresponding momentum transformation

$$E' = \alpha_R \gamma (E - up_x), \quad p'_x = \alpha_R \gamma (p_x - uE/c^2), \quad p'_y = \alpha_R y, \quad p'_z = \alpha_R z \quad (6)$$

with which the four dimensional contraction $p_\mu x^\mu$ is an invariant, allowing then a coherent description of plane waves. Here c and R are invariant and in the limit $R \rightarrow \infty$, the above transformations reduce to the Lorentz transformation for the coordinates as well as for the energy-momentum vector.

The R -algebra, constituted by (1), (2) and (3), is completed in [8] by involving pure rotation generators,

$$M_i = \frac{1}{2} \epsilon_{ijk} J_{jk}, \quad (7)$$

and boost ones,

$$\tilde{N}_i = J_{0i}, \quad (8)$$

where $J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu$ stands for the angular momentum, $i, j, \dots = 1, 2, 3$ and ϵ_{ijk} is the Levi-Civita antisymmetric tensor ($\epsilon_{123} = 1$). Following the Magpantay approach [9], we proposed a modified expression of the boost generators

$$N_i = J_{0i} - \frac{1}{2R} \eta_{\mu\nu} x^\mu x^\nu p_i, \quad (9)$$

that allows to construct the first Casimir invariant of the theory.

The quantization was done by the substitution of x^μ and p^μ by the corresponding operators and the Poisson brackets by commutators. The resulting phase space algebra of the R -Minkowski spacetime

$$[x^\mu, x^\nu] = 0, \quad (10)$$

$$[x^\mu, p^\nu] = -i\hbar(\eta^{\mu\nu} - \frac{1}{R}\eta^{0\nu}x^\mu), \quad (11)$$

$$[p^\mu, p^\nu] = -\frac{i\hbar}{R}(p^\mu\eta^{0\nu} - p^\nu\eta^{\mu 0}), \quad (12)$$

and the underlying R -Poincaré algebra

$$[N_i, p_0] = -i\hbar p_i + i\hbar \frac{1}{R} N_i, \quad (13)$$

$$[N_i, p_j] = -i\hbar \delta_{ij} p_0 - i\hbar \frac{1}{R} \epsilon_{ijk} M_k, \quad (14)$$

$$[M_i, p_0] = 0, \quad (15)$$

$$[M_i, p_j] = i\hbar \epsilon_{ijk} p_k, \quad (16)$$

$$[M_i, M_j] = i\hbar \epsilon_{ijk} M_k, \quad (17)$$

$$[M_i, N_j] = i\hbar \epsilon_{ijk} N_k, \quad (18)$$

$$[N_i, N_j] = -i\hbar \epsilon_{ijk} M_k, \quad (19)$$

allowed us to obtain the following expression for the first Casimir

$$C = p_0^2 - p^i p^i + \frac{1}{R}(N^i p^i + p^i N^i) - \frac{1}{R^2} M^i M^i, \quad (20)$$

which obviously reduces, in the limit $R \rightarrow \infty$, to the first Poincaré Casimir.

In the present work, we will focus in the R -Minkowski spacetime on the Dirac equation which is already investigated in the context of the DSR [10, 11, 12]. Also, we aim to explore more the correspondence, already established in [8], between de Sitter spacetime and the R -Minkowski spacetime. This may offer more possibilities to handle the issue of constructing physical observables in the de Sitter space [13, 14]. Furthermore, we propose a new approach for solving the free Dirac equation in the de Sitter spacetime by using the Schrodinger picture established in this context by Cotăescu [15].

The paper is organized as follows. In section 2, we establish the free Dirac equation in the R -Minkowski spacetime and show that the obtained result is identically the Dirac equation in a conformally flat de Sitter spacetime. In section 3, we construct the Schrödinger picture version of this equation and in section 4, we propose within this picture a new procedure to solve the free Dirac equation in de Sitter spacetime. In section 5, we give some concluding remarks.

2 Dirac equation in R -Minkowski spacetime

In [8], we proposed the following representation for the momentum

$$p^0 = i\hbar(\partial^0 - \frac{1}{R}x^\mu\partial_\mu), \quad (21)$$

$$p^i = i\hbar\partial^i, \quad (22)$$

with which the complete algebra (10)-(19) is satisfied, and therefore showed that expression (20) of the Casimir is exactly the Klein-Gordon operator of a conformal flat metric of de Sitter spacetime. This result established a correspondence between R -Minkowski spacetime and de Sitter spacetime.

For constructing the Dirac equation in the R -Minkowski spacetime, we impose to the square of Dirac operator to reproduce partly Klein-Gordon operator given by expression (20) of the Casimir. In view of this, let us express C only in terms of the operators x and p . With the use of expressions (7) and (9) and taking into account commutators (10), (11) and (12), we can show that

$$M^i M^i = \epsilon^{ijk} \epsilon^{ilm} x^j p^k x^l p^m = \vec{x}^2 \vec{p}^2 - (\vec{x} \cdot \vec{p})^2 + i\hbar \vec{x} \cdot \vec{p}, \quad (23)$$

$$N^i p^i = x^0 \vec{p}^2 - \vec{x} \cdot \vec{p} p^0 - \frac{i\hbar}{R} \vec{x} \cdot \vec{p} - \frac{1}{2R} x^{02} \vec{p}^2 + \frac{1}{2R} \vec{x}^2 p^2, \quad (24)$$

$$p^i N^i = x^0 \vec{p}^2 - \vec{x} \cdot \vec{p} p^0 - \frac{i\hbar}{R} \vec{x} \cdot \vec{p} - \frac{1}{2R} x^{02} \vec{p}^2 + \frac{1}{2R} \vec{x}^2 p^2 + 3i\hbar p_0. \quad (25)$$

The above vectors are three-dimensional ones. Substituting these relations in (20), we obtain

$$C = p_0^2 - \left(1 - \frac{x^0}{R}\right)^2 \vec{p}^2 - \frac{2}{R} \vec{x} \cdot \vec{p} p^0 + \frac{1}{R^2} (\vec{x} \cdot \vec{p})^2 - \frac{3i\hbar}{R^2} \vec{x} \cdot \vec{p} + \frac{3i\hbar}{R} p_0. \quad (26)$$

It is interesting to remark that by using this last expression and the fact that

$$\gamma^0 \gamma^i p_0 \left(1 - \frac{x^0}{R}\right) p_i + \gamma^i \gamma^0 \left(1 - \frac{x^0}{R}\right) p_i p_0 = 0 \quad (27)$$

and

$$\frac{\gamma^i \gamma^0}{R} p_i x^j p_j + \frac{\gamma^0 \gamma^i}{R} x^j p_j p_i = -i\hbar \frac{\gamma^0 \gamma^i}{R} p_i, \quad (28)$$

we can show

$$\left[\gamma^0 p_0 + \left(1 - \frac{x^0}{R}\right) \gamma^i p_i + \frac{\gamma^0}{R} x^i p_i + i \frac{3\hbar \gamma^0}{2R} + mc \right] \left[\gamma^0 p_0 + \left(1 - \frac{x^0}{R}\right) \gamma^i p_i + \frac{\gamma^0}{R} x^i p_i + i \frac{3\hbar \gamma^0}{2R} - mc \right] = C - m^2 c^2 - i\hbar \left(1 - \frac{x^0}{R}\right) \frac{\gamma^0 \gamma^i}{R} p_i - \frac{9\hbar^2}{4R^2}, \quad (29)$$

γ^0 and γ^i being Dirac matrices. Equation (29) suggests to write the Dirac equation in R -Minkowski spacetime in the following form

$$\left[\gamma^0 p_0 + \left(1 - \frac{x^0}{R}\right) \gamma^i p_i + \frac{\gamma^0}{R} x^i p_i + i \frac{3\gamma^0}{2R} - mc \right] \Psi = 0. \quad (30)$$

Indeed, the last two terms in (29) that make the square of the Dirac operator different from the Klein-Gordon operator are the manifestation of the spin 1/2 for the Dirac particle. In fact, in a curved spacetime, the solution to the generally covariant Dirac equation is not a solution to the generally covariant Klein-Gordon equation but to the generally covariant Pauli-Schrödinger equation describing spin 1/2 particles in a gravitational field [16, 17]. In the square of the spinor covariant derivative, the spinorial nature of the Dirac particle appears within the terms involving Fock-Ivanenko coefficients. For more details, the presence of these two additional terms is justified in Appendix A.

Using the representation given by (21) and (22), proposed in the R -Poincaré algebra context [8], we obtain the differential form of (30)

$$\left[i \left(1 - \frac{x^0}{R}\right) \gamma^0 \partial_0 + i \left(1 - \frac{x^0}{R}\right) \gamma^i \partial_i + i \frac{3\gamma^0}{2R} - mc/\hbar \right] \Psi = 0. \quad (31)$$

This is exactly the Dirac equation in the conformally flat de Sitter spacetime with a conformal factor $a(t) = (1 - x^0/R)^{-1}$. Again, a correspondence between R -Minkowski spacetime and de Sitter spacetime is established.

3 The Schrödinger picture

In order to solve equation (31), we will use the Schrödinger picture developed in [15]. We will give an exact solution of the Dirac equation in the R -Minkowski or de Sitter spacetime, that transforms to the form found by Shishkin [18] and Cotaescu [19] in the natural picture.

Cotaescu showed that the transformation $\Psi(x) \rightarrow \Psi_S(x) = W(x)\Psi(x)$ leading to the Schrödinger picture is produced by the operator of time dependent dilatations

$$W(x) = \exp \left[-\ln(\alpha(t))(\vec{x} \cdot \vec{\partial}) \right], \quad (32)$$

with the following properties:

$$W(x)^\dagger = \alpha^3(t) W(x)^{-1}, \quad (33)$$

and

$$W(x)F(x)W(x)^{-1} = F\left(\frac{1}{\alpha(t)}x\right), \quad W(x)G(\vec{\partial})W(x)^{-1} = G\left(\alpha(t)\vec{\partial}\right), \quad (34)$$

F and G being arbitrary functions. Setting $\alpha \equiv \xi^{-1}$, in the R -Minkowski spacetime, we have $\xi = 1 - x^0/R$ and then

$$W(x) = \exp\left[\frac{i}{\hbar} \ln(\xi)(\vec{x} \cdot \vec{p})\right] \quad (35)$$

and

$$W(x)G(\vec{p})W(x)^{-1} = G(\alpha(t)\vec{p}). \quad (36)$$

Using the fact that p_0 commute with $\vec{x} \cdot \vec{p}$ and $[p_0, \xi] = -i\hbar\xi/R$, we have

$$\begin{aligned} W(x)p_0W(x)^{-1} &= W(x)[p_0, W(x)^{-1}] + p_0 \\ &= W(x)[p_0, \xi] \frac{\partial W(x)^{-1}}{\partial \xi} + p_0 \\ &= p_0 - \frac{\vec{x} \cdot \vec{p}}{R}. \end{aligned} \quad (37)$$

By substituting in (30) $\Psi(x)$ by $W^{-1}\Psi_S(x)$ and multiplying at left the same equation by W , we obtain in the Schrödinger picture the free Dirac equation in the R -Minkowski spacetime

$$\left[\gamma^0 p_0 + \gamma^i p_i - 2\frac{\gamma^0}{R}\vec{x} \cdot \vec{p} + i\frac{3\gamma^0}{2R} - mc\right] \Psi_S = 0, \quad (38)$$

where relations (36) and (37) have been used. With the representation given in (21) and (22), the differential form of the last equation is

$$\left[i\gamma^0 \left(\left(1 - \frac{x^0}{R}\right)\partial_0 + \frac{x^i \partial_i}{R} + \frac{3}{2R}\right) + i\gamma^i \partial_i - mc/\hbar\right] \Psi_S = 0. \quad (39)$$

4 The solution

Making the substitution $\Psi_S = (1 - x^0/R)^{3/2}\tilde{\Psi}$ in equation (39), we obtain

$$\left[i\gamma^0 \left(\left(1 - \frac{x^0}{R}\right)\partial_0 + \frac{x^i \partial_i}{R}\right) + i\gamma^i \partial_i - mc/\hbar\right] \tilde{\Psi} = 0. \quad (40)$$

We put in what follows $c = \hbar = 1$. Multiplying at left (40) by γ^0

$$\left[i\left(1 - \frac{x^0}{R}\right)\partial_0 + i\left(\alpha^i + \frac{x^i}{R}\right)\partial_i - \gamma^0 m\right] \tilde{\Psi} = 0. \quad (41)$$

and using the variable ξ , we obtain

$$[\xi \partial_\xi - (R\alpha^i + x^i)\partial_i - i\gamma^0 Rm] \tilde{\Psi} = 0. \quad (42)$$

The function $\tilde{\Psi}$ must be a bispinor

$$\tilde{\Psi} = \begin{pmatrix} \varphi \\ \chi \end{pmatrix} \quad (43)$$

where φ and χ are two spinors to be determined. Using the standard Dirac representation for γ^μ matrices, equation (42) leads to the following system

$$(\xi\partial_\xi - x^i\partial_i - iRm)\varphi - R\sigma^i\partial_i\chi = 0, \quad (44)$$

$$(\xi\partial_\xi - x^i\partial_i + iRm)\chi - R\sigma^i\partial_i\varphi = 0. \quad (45)$$

Multiplying at left (45) by $R\sigma^i\partial_i$ and using (44), we get to

$$(\xi^2\partial_\xi^2 - 2\xi x^i\partial_i\partial_\xi + 2x^i\partial_i + x^ix^j\partial_i\partial_j - R^2\nabla^2 + iRm + R^2m^2)\varphi = 0, \quad (46)$$

which we can put in the form

$$[(\xi\partial_\xi - x^i\partial_i)^2 - (\xi\partial_\xi - x^i\partial_i) - R^2\nabla^2 + iRm + R^2m^2]\varphi = 0. \quad (47)$$

In the spherical coordinate system, (r, θ, ϕ) , ∇^2 can be separated as

$$\nabla^2 = \nabla_r^2 + \frac{1}{r^2}\nabla_{(\theta,\phi)}^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{\vec{L}^2}{r^2}, \quad (48)$$

where $\vec{L} = \vec{x} \times \vec{p}$ is the angular momentum generator

$$\vec{L}^2 = -\nabla_{(\theta,\phi)}^2 = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}. \quad (49)$$

Knowing that $x^i\partial_i = r\partial_r$, equation (47) becomes

$$\left[(\xi\partial_\xi - r\partial_r)^2 - (\xi\partial_\xi - r\partial_r) - \frac{R^2}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{R^2}{r^2}\vec{L}^2 + iRm + R^2m^2 \right]\varphi = 0. \quad (50)$$

This equation is separable and gives a Sturm-Liouville system. So, we can consider the following separation scheme

$$\varphi = U(\xi, r)\Omega(\theta, \phi), \quad (51)$$

which leads to the following system

$$\vec{L}^2\Omega(\theta, \phi) - \lambda\Omega(\theta, \phi) = 0, \quad (52)$$

$$\left[(\xi\partial_\xi - r\partial_r)^2 - (\xi\partial_\xi - r\partial_r) - \frac{R^2}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + iRm + R^2m^2 + \frac{R^2}{r^2}\lambda \right] U(\xi, r) = 0, \quad (53)$$

where λ is a separation constant. The solution of (52) is given for $\lambda = l(l+1)$ by the spherical harmonic spinors, constituted of the usual spherical harmonics $Y_{lm}(\theta, \phi)$ and the base spinors $\chi(s_3)$,

$$\Omega(\theta, \phi) \equiv \Omega_{j,m}^l(\theta, \phi) = \langle j, m \mid l, m'; \frac{1}{2}, s_3 \rangle Y_{lm'}(\theta, \phi)\chi(s_3), \quad (54)$$

where $\langle j, m \mid l, m'; \frac{1}{2}, s_3 \rangle$ are the Clebsch-Gordon coefficients. Explicitly, these spherical harmonic spinors are given by [20]

$$\Omega_{j,m}^l(\theta, \phi) = \frac{1}{\sqrt{2j}} \begin{pmatrix} \sqrt{j+m} Y_{l,m-\frac{1}{2}} \\ \sqrt{j-m} Y_{l,m+\frac{1}{2}} \end{pmatrix}, \quad \text{for } j = l + \frac{1}{2} \quad (55)$$

$$\Omega_{j,m}^l(\theta, \phi) = \frac{1}{\sqrt{2j+2}} \begin{pmatrix} -\sqrt{j-m+1} Y_{l,m-\frac{1}{2}} \\ \sqrt{j+m+1} Y_{l,m+\frac{1}{2}} \end{pmatrix}, \quad \text{for } j = l - \frac{1}{2}. \quad (56)$$

Concerning equation (53), if we use the following variable change

$$\eta = \xi r, \quad \zeta = \xi, \quad (57)$$

it will take the following separable form

$$\begin{aligned} \frac{1}{R^2 \zeta^2} (\zeta^2 \partial_\zeta^2 + i R m + R^2 m^2) U_l(\zeta, \eta) \\ = \left(\frac{\partial^2}{\partial \eta^2} + \frac{2}{\eta} \frac{\partial}{\partial \eta} - \frac{l(l+1)}{\eta^2} \right) U_l(\zeta, \eta). \end{aligned} \quad (58)$$

Thus, if we set $U_l(\zeta, \eta) = V(\zeta) Z_l(\eta)$, we can obtain the following system

$$(\zeta^2 \partial_\zeta^2 + R^2 \bar{\kappa}^2 \zeta^2 + i R m + R^2 m^2) V(\zeta) = 0, \quad (59)$$

$$\left[\eta^2 \frac{\partial^2}{\partial \eta^2} + 2\eta \frac{\partial}{\partial \eta} + \bar{\kappa}^2 \eta^2 - l(l+1) \right] Z_l(\eta) = 0, \quad (60)$$

where $\bar{\kappa}^2$ is a separation constant. Relation (59) is a Bessel's equation. Its solution can be expressed in term of Hankel function of first kind [21, 22]

$$V(\zeta) = \zeta^{\frac{1}{2}} H_{\nu_-}(R \bar{\kappa} \zeta), \quad (61)$$

where $\nu_- = 1/2 - i R m$. The general solution of Dirac's equation with ν_- is a linear combination that contains terms with each of the above solutions. The other solution with $-\nu_-$ is obtained with the same manner. Relation (60) is also a Bessel's equation. Its solutions are the spherical Bessel functions: $j_l(\bar{\kappa} \eta) = j_l(\bar{\kappa} r \xi)$, $y_l(\bar{\kappa} \eta) = y_l(\bar{\kappa} r \xi)$ and $h_l^{1,2}(\bar{\kappa} \eta) = h_l^{1,2}(\bar{\kappa} r \xi)$. For $|\bar{\kappa}| < 0$, there is no solution that is bounded at infinity and regular at the origin. For $|\bar{\kappa}| > 0$, the only solution that is bounded everywhere is

$$Z(\xi, r) = N_l j_l(\bar{\kappa} r \xi). \quad (62)$$

where N_l stands for a normalization constant. Thus, the physical solution to (53) is

$$U_l(\xi, r) = N_l \xi^{\frac{1}{2}} H_{\nu_-}(R \bar{\kappa} \xi) j_l(\bar{\kappa} r \xi). \quad (63)$$

Requiring that this solution be reduced in the limit $R \rightarrow \infty$ to the usual one of Dirac's equation in the Minkowski spacetime imposes $\bar{\kappa} = p$, where p is the four-momentum [23]. It follows that

$$U_l(\xi, r) = N_l \xi^{\frac{1}{2}} H_{\nu_-}(R p \xi) j_l(p r \xi). \quad (64)$$

It remains to determine the second spinor χ . Introducing $X = p r \xi$ and using the property

$$\vec{\sigma} \cdot \vec{\nabla} f(r) \Omega_{j,m}^l = \left[\frac{df(r)}{dr} + \frac{1+\kappa}{r} f(r) \right] (\vec{\sigma} \cdot \hat{r}) \Omega_{j,m}^l, \quad (65)$$

where

$$\kappa = \mp(j + \frac{1}{2}) = \begin{cases} -(l+1) & \text{for } j = l + \frac{1}{2} \\ l & \text{for } j = l - \frac{1}{2} \end{cases} \quad \text{and} \quad (\vec{\sigma} \cdot \hat{r}) \Omega_{j,m}^{\kappa} = -\Omega_{j,m}^{-\kappa}, \quad (66)$$

we have

$$\begin{aligned} R\sigma^i \partial_i \varphi &= N_l R \sigma^i \partial_i [j_l(X) \Omega_{j,m}^l(\theta, \phi)] \xi^{\frac{1}{2}} H_{\nu_-}(Rp\xi) \\ &= N_l R \left\{ \left[\frac{dj_l(X)}{dr} + \frac{1+\kappa}{r} j_l(X) \right] (\vec{\sigma} \cdot \hat{r}) \Omega_{j,m}^l \right\} \xi^{\frac{1}{2}} H_{\nu_-}(Rp\xi) \\ &= -N_l Rp \xi^{\frac{3}{2}} H_{\nu_-}(Rp\xi) j_{\bar{l}}(X) \Omega_{j,m}^{\bar{l}}, \end{aligned} \quad (67)$$

where we have used the properties of Bessel's functions of integer order [21, 22]

$$\frac{dj_l}{d\rho}(\rho) = \frac{l}{\rho} j_l(\rho) - j_{l+1}(\rho), \quad \frac{dj_l}{d\rho}(\rho) = j_{l-1}(\rho) - \frac{l+1}{\rho} j_l(\rho), \quad j_{-l} = (-1)^l j_l,$$

and we have set $\bar{l} = 2j - l$. With the use of (67) and of the fact that $x^i \partial_i = r \partial_r$, equation (45) gives

$$(\xi \partial_\xi - r \partial_r + iRm) \chi = -N_l Rp \xi^{\frac{3}{2}} H_{\nu_-}(Rp\xi) j_{\bar{l}}(X) \Omega_{j,m}^{\bar{l}}. \quad (68)$$

Introducing $\tilde{\chi} = \xi^{-1/2} \chi$, the last equation takes the form

$$\left(\xi \partial_\xi - r \partial_r + iRm + \frac{1}{2} \right) \tilde{\chi} = -N_l Rp \xi H_{\nu_-}(Rp\xi) j_{\bar{l}}(X) \Omega_{j,m}^{\bar{l}}. \quad (69)$$

Setting $z = Rp\xi$, we get to the following equation:

$$\left(z \partial_z - r \partial_r + iRm + \frac{1}{2} \right) \tilde{\chi} = -N_l z H_{\frac{1}{2} - iRm}(Rp\xi) j_{\bar{l}}(X) \Omega_{j,m}^{\bar{l}}. \quad (70)$$

As $X = pr\xi$, it is easy to check that $(\xi \partial_\xi - r \partial_r) j_l(X) = 0$. It follows that by using the recurrence relation for Bessel's functions

$$z \partial_z \mathcal{C}_\nu(z) + \nu \mathcal{C}_\nu(z) = z \mathcal{C}_{\nu-1}(z), \quad (71)$$

where $\mathcal{C} \equiv J, Y, H$, and the properties of Hankel's functions of first kind

$$H_{-\nu}^{(1)}(z) = e^{i\pi\nu} H_\nu^{(1)}(z), \quad H_{-\nu}^{(2)}(z) = e^{-i\pi\nu} H_\nu^{(2)}(z), \quad (72)$$

the solution for the spinor χ can be obtained as

$$\chi = -N_l e^{-i\pi\nu_-} H_{\nu_+}^{(1)}(Rp\xi) j_{\bar{l}}(pr\xi) \Omega_{j,m}^{\bar{l}}, \quad (73)$$

where $\nu_+ = 1/2 + iRm$. Then, the solution for Dirac's equation in its reduced form (40) can be written as

$$\tilde{\Psi} = N_l \xi^{\frac{1}{2}} e^{-\frac{\pi}{2} Rm} \begin{pmatrix} e^{\frac{\pi}{2} Rm} H_{\nu_-}^{(1)}(Rp\xi) j_l(pr\xi) \Omega_{j,m}^l(\theta, \phi) \\ i e^{-\frac{\pi}{2} Rm} H_{\nu_+}^{(1)}(Rp\xi) j_{\bar{l}}(pr\xi) \Omega_{j,m}^{\bar{l}}(\theta, \phi) \end{pmatrix}. \quad (74)$$

Finally, one can write the solution of the Dirac equation, relation (39), as

$$\Psi_S = \xi^{\frac{3}{2}} \tilde{\Psi} = N_l \xi^2 e^{-\frac{\pi}{2} Rm} \begin{pmatrix} e^{\frac{\pi}{2} Rm} H_{\nu_-}^{(1)}(Rp\xi) j_l(pr\xi) \Omega_{j,m}^l(\theta, \phi) \\ i e^{-\frac{\pi}{2} Rm} H_{\nu_+}^{(1)}(Rp\xi) j_{\bar{l}}(pr\xi) \Omega_{j,m}^{\bar{l}}(\theta, \phi) \end{pmatrix}. \quad (75)$$

Let us now to determine the normalization constant N_l by using the condition

$$\int_{\Sigma} d^3x \Psi^\dagger(x) \Psi(x) = 1, \quad (76)$$

where $d^3x = r^2 dr d\Omega$ and the integration must be done over the spacelike hypersurface $\Sigma \equiv x^0 = cst$. For more details, see Appendix B. The standard normalization condition of the spherical Bessel's function being [24]

$$\int_0^\infty r^2 j_l(kr) j_{l'}(k'r) dr = \frac{\pi}{2k^2} \delta(k - k') \delta_{ll'}, \quad (77)$$

for $k = p\xi$, we have

$$\int_0^\infty r^2 j_l(p\xi r) j_{l'}(p\xi' r) dr = \frac{\pi}{2p^2 \xi^3} \delta(p - p') \delta_{ll'}, \quad (78)$$

and since the spherical harmonic spinors are normalized with respect to the relation

$$\int (\Omega^*)_{j',m'}^{l'} \Omega_{j,m}^l d\Omega = \delta_{jj'} \delta_{ll'} \delta_{mm'}, \quad (79)$$

it follows that

$$|N_l|^2 \frac{\pi}{2p^2 \xi^3} \xi^4 e^{-\pi Rm} \left[e^{-\pi Rm} (H_{\nu_+}^{(1)})^* (Rp\xi) H_{\nu_+}^{(1)} (Rp\xi) + e^{\pi Rm} (H_{\nu_-}^{(1)})^* (Rp\xi) H_{\nu_-}^{(1)} (Rp\xi) \right] = 1. \quad (80)$$

By using the following properties of the Hankel functions of first kind

$$(H_{\nu_\pm}^{(1,2)})^* = H_{\nu_\mp}^{(2,1)}, \quad \nu_\pm = \frac{1}{2} \pm k, \quad (81)$$

$$e^{\pm \pi k} H_{\nu_\mp}^{(1)}(z) H_{\nu_\pm}^{(2)}(z) + e^{\mp \pi k} H_{\nu_\pm}^{(1)}(z) H_{\nu_\mp}^{(2)}(z) = \frac{4}{\pi z}, \quad (82)$$

equation (80) allows to deduce that

$$|N_l| = \frac{R^{\frac{1}{2}} p^{\frac{3}{2}}}{2} e^{\frac{1}{2} \pi Rm}. \quad (83)$$

Finally, the normalized wave function reads

$$\Psi_S = \frac{p}{2} \sqrt{\frac{\pi R}{2\xi r}} \xi^2 \begin{pmatrix} e^{\frac{\pi}{2} Rm} H_{\nu_-}^{(1)}(Rp\xi) j_l(pr) \Omega_{j,m}^l(\theta, \phi) \\ i e^{-\frac{\pi}{2} Rm} H_{\nu_+}^{(1)}(Rp\xi) j_{\bar{l}}(pr) \Omega_{j,m}^{\bar{l}}(\theta, \phi) \end{pmatrix} \quad (84)$$

Using the operator W given by (35), one can easily obtain the free Dirac equation solution Ψ_{NP} in the natural picture

$$\Psi_{NP} = W^{-1} \Psi_S = \frac{p}{2} \sqrt{\frac{\pi R}{2r}} \xi^2 \begin{pmatrix} e^{\frac{\pi}{2} Rm} H_{\nu_-}^{(1)}(Rp\xi) j_l(pr) \Omega_{j,m}^l(\theta, \phi) \\ i e^{-\frac{\pi}{2} Rm} H_{\nu_+}^{(1)}(Rp\xi) j_{\bar{l}}(pr) \Omega_{j,m}^{\bar{l}}(\theta, \phi) \end{pmatrix} \quad (85)$$

If we perform a transition to the moving chart $\{\tau, \vec{x}\}$ with the standard flat metric $ds^2 = c^2 d\tau^2 - e^{2\omega\tau} d\vec{x}^2$ with $\xi = e^{-\omega\tau}$ and $\omega = 1/R$, one can easily check that Ψ_{NP} is exactly the same solution found by Shishkin [18] and by Cotăescu & al. [19].

5 Conclusion

In the present work, we constructed the free Dirac equation in the R -Minkowski spacetime. After using a certain realization of the R -Poincaré algebra, it turned out that the obtained equation is exactly the Dirac equation in the conformally flat de Sitter spacetime. This is a further proof of the correspondence between the R -Minkowski and the de Sitter space such that the physics of R -Poincaré algebra is the same as in the de Sitter relativity. So, this correspondence could be used to construct well-defined physical observables in the de Sitter spacetime.

We also presented a new method for solving Dirac equation in the conformally flat patch of de Sitter spacetime within the Schrödinger picture. The latter was introduced by Cotăescu to investigate Dirac and Klein-Gordon equations in the context of de Sitter spacetime.

Appendix A: Squared Dirac equation in curved spacetime

In a curved spacetime, Dirac's equation is given by [25]

$$\left(i\gamma^\mu D_\mu - \frac{mc}{\hbar}\right)\psi(x) = 0, \quad (86)$$

where D_μ is the spinor covariant derivative,

$$D_\mu\psi = (\partial_\mu + \Omega_\mu)\psi, \quad (87)$$

and

$$\Omega_\mu(x) \equiv -\frac{i}{4}\omega_{ab\mu}(x)\sigma^{ab} = \frac{1}{8}\omega_{ab\mu}(x)[\gamma^a, \gamma^b] \quad (88)$$

$$\omega^a{}_{b\mu} = e_\nu{}^a(\partial_\mu e_b{}^\nu + e_b{}^\sigma\Gamma_{\sigma\mu}^\nu), \quad (89)$$

are respectively the Fock-Ivanenko coefficients[26] and the spin connection, $\Gamma_{\sigma\mu}^\nu$ being the Christoffel symbols. It follows that the square of Dirac's equation [16]

$$(i\gamma^\mu D_\mu + mc/\hbar)(i\gamma^\nu D_\nu - mc/\hbar)\psi = 0 \quad (90)$$

can be rewritten in the form

$$\left[g^{\mu\nu}D_\mu D_\nu - \frac{1}{2}\sigma^{\mu\nu}K_{\mu\nu} + (mc/\hbar)^2\right]\psi = 0, \quad (91)$$

where

$$K_{\mu\nu} \equiv \frac{1}{2}(D_\nu D_\mu - D_\mu D_\nu) = \partial_\nu\Omega_\mu - \partial_\mu\Omega_\nu + [\Omega_\nu, \Omega_\mu] \quad (92)$$

is the spin curvature. Equation (91) is the generally covariant extension of the Pauli-Schrödinger equation that describes spin 1/2 particles in a gravitational field [16, 17]. The first term contains, in addition to the Klein-Gordon operator, terms involving the Fock-Ivanenko coefficients Ω_μ

$$\begin{aligned} g^{\mu\nu}D_\mu D_\nu\psi &= (g^{\mu\nu}\partial_\mu\partial_\nu - g^{\mu\nu}\Gamma_{\mu\nu}^\lambda\partial_\lambda)\psi + g^{\mu\nu}[(D_\mu\Omega_\nu) + 2\Omega_\nu\partial_\mu]\psi \\ &= \square_{KG}\psi + g^{\mu\nu}[(D_\mu\Omega_\nu) + 2\Omega_\nu\partial_\mu]\psi. \end{aligned} \quad (93)$$

In our case, the R -Minkowski space corresponds to the de Sitter spacetime given by the metric

$$ds^2 = \frac{1}{(1 - x^0/R)^2} [(dx^0)^2 - d\vec{x}^2] = \frac{1}{\xi^2} [(dx^0)^2 - d\vec{x}^2]. \quad (94)$$

In the chart with $x^0 \in (-\infty, 0]$, the tetrad field is given by $e_a^\mu = \xi \delta_a^\mu$ and the spin connection by $\omega_{\mu ab} = (R\xi)^{-1} [\eta_{\mu b} \delta_a^0 - \eta_{\mu a} \delta_b^0]$. Then, the covariant derivative reads

$$D_\mu = \partial_\mu + \frac{1}{4R\xi} \eta_{\mu a} [\gamma^0, \gamma^a], \quad (95)$$

and a simple calculus gives for the sum of the terms involving the Fock-Ivanenko coefficients Ω

$$\begin{aligned} g^{\mu\nu} [(D_\mu \Omega_\nu) + 2\Omega_\nu \partial_\mu] &= g^{\mu\nu} [(\partial_\mu \Omega_\nu) - \Gamma_{\nu\mu}^\sigma \Omega_\sigma + \Omega_\mu \Omega_\nu + 2\Omega_\nu \partial_\mu] \\ &= -\frac{1}{R} \xi \gamma^0 \gamma^i \partial_i - \frac{3}{4R^2}. \end{aligned} \quad (96)$$

On the other hand, it is known that

$$-\frac{1}{2} \sigma^{\mu\nu} K_{\mu\nu} = \frac{\mathcal{R}}{4}, \quad (97)$$

where \mathcal{R} is the Ricci scalar and that in the de Sitter space, we have $\mathcal{R} = 12/R^2$. Taking into account equations (93), (96) and (97), relation (91) can be rewritten as

$$\left[\square_{KG} - \frac{1}{R} \xi \gamma^0 \gamma^i \partial_i + \frac{9}{4R^2} + (mc/\hbar)^2 \right] \psi = 0. \quad (98)$$

Comparing to (29), it is clear that expressions of terms which make the square of Dirac's operator different from the one of Klein-Gordon are established.

Appendix B: Remark on the integration measure

In the natural representation (NP), the normalization of the wave function, expressed through the canonical variable $X^\mu = x^\mu/\xi$, is given by the usual condition in special relativity

$$\begin{aligned} \langle \Psi_{NP}, \Psi_{NP} \rangle &= \int_\Sigma d^3 X \bar{\Psi}_{NP}(X) \gamma^0 \Psi_{NP}(X) \\ &= \int_\Sigma d^3 X \Psi_{NP}^\dagger(X) \Psi_{NP}(X) = 1, \end{aligned} \quad (99)$$

where the integration must be done over the spacelike hypersurface Σ , determined by $X^0 = cst$ meaning that $\xi = cst$. So, the spacelike hypersurface at a constant time in the R -Minkowski spacetime is given by

$$d^3 X|_{T=cst} = dX \wedge dY \wedge dZ|_{t=cst} = d^3 x \xi^{-3}. \quad (100)$$

Thus, the normalization condition in the R -Minkowski spacetime is given by

$$\langle \Psi_{NP}, \Psi_{NP} \rangle = \int_\Sigma d^3 x \xi^{-3} \Psi_{NP}^\dagger(x) \Psi_{NP}(x) = 1, \quad (101)$$

where $d^3 x = r^2 dr d\Omega$. Taking into account relation (33), one can check that in the R -Minkowski spacetime the wave function, $\Psi \equiv \Psi_{SP}$, in the Schrödinger

representation, is normalized with respect to the usual condition of special relativity

$$\begin{aligned}
\langle \Psi_{SP}, \Psi_{SP} \rangle &= \langle \Psi_{NP}, \Psi_{NP} \rangle \\
&= \int_{\Sigma} d^3x \xi^{-3} \Psi_{NP}^{\dagger}(x) \Psi_{NP}(x) \\
&= \int_{\Sigma} d^3x \Psi_{SP}^{\dagger}(x) \Psi_{SP}(x) = 1.
\end{aligned} \tag{102}$$

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